

FRACTIONAL TIME STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we introduce a class of stochastic partial differential equations (SPDEs) with fractional time-derivatives, and study the L_2 -theory of the equations. This class of SPDEs can be used to describe random effects on transport of particles in medium with thermal memory or particles subject to sticking and trapping.

1. Introduction

Fractional calculus has attracted lots of attentions in several fields including mathematics, physics, chemistry, engineering, hydrology and even finance and social sciences. The classical heat equation $\partial_t u = \Delta u$ describes heat propagation in homogeneous medium. The time-fractional diffusion equation $\partial_t^\beta u = \Delta u$ with $0 < \beta < 1$ has been widely used to model the anomalous diffusions exhibiting subdiffusive behavior, due to particle sticking and trapping phenomena. Here the fractional time derivative ∂_t^β is the Caputo derivative of order $\beta \in (0, 1)$, defined by

$$\frac{\partial^\beta f(t)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta} (f(s) - f(0)) ds \quad (1.1)$$

where Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$.

Fractional diffusion equations are becoming popular in many areas of application [3, 4, 7, 9, 10, 11, 12]. So far, on the basis of either deterministic or probabilistic methods, the study of fractional calculus is mainly restricted to deterministic equations; see [2, 8, 10, 14] and the references therein. In this paper, we introduce and investigate a class of stochastic partial differential equations (SPDEs) with fractional time derivatives.

The SPDEs with fractional time derivative that we are going to study in this paper naturally arise from the consideration of the heat equation in a material of thermal memory. Let $u(t, x)$, $e(t, x)$ and $\vec{F}(t, x)$ denote the body temperature, internal energy and flux density, respectively. Then the relations

$$\frac{\partial e}{\partial t}(t, x) = -\operatorname{div} \vec{F}, \quad (1.2)$$

$$e(t, x) = \beta u(t, x), \quad \vec{F}(t, x) = -\lambda \nabla u(t, x), \quad \beta, \lambda > 0$$

yield the classical heat equation $\beta \frac{\partial u}{\partial t} = \lambda \Delta u$. According to the law of the classical heat equation, the speed of the heat flow is infinite. However in real modeling, the propagation speed can be finite

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because the heat flow can be disrupted by the response of the material. It has been proved (see e.g., [12, 7]) that in a material with thermal memory

$$e(t, x) = \bar{\beta}u(t, x) + \int_0^t n(t-s)u(s, x)ds \quad (1.3)$$

holds with some appropriate constant $\bar{\beta}$ and kernel n . Typically, $n(t)$ is a positive decreasing function which blows up near $t = 0$, for instance $n(t) = t^{-\alpha}$ for $\alpha \in (0, 1)$. In this case the convolution implies that nearer past affects the present more. If in addition the internal energy $e(t, x)$ depends also on past random effects, then (1.3) becomes

$$e(t, x) = \bar{\beta}u(t, x) + \int_0^t n(t-s)u(s, x)ds + \int_0^t \ell(t-s)h(s, u(s, x))dW_s, \quad (1.4)$$

where W is a random process, such as Brownian motion, modeling the random effects. If $u(0, x) = 0$, $\bar{\beta} = 0$, $n(t) = \Gamma(1 - \beta_1)^{-1}t^{-\beta_1}$ and $\ell(t) = \Gamma(2 - \beta_2)^{-1}t^{1-\beta_2}$ for some constants $\beta_i \in (0, 1)$, then (1.4) (after differentiation in t) becomes

$$\begin{aligned} -\operatorname{div} \vec{F} = \frac{\partial e}{\partial t}(t, x) &= \frac{1}{\Gamma(1 - \beta_1)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\beta_1} u(s, x)ds \\ &\quad + \frac{1}{\Gamma(2 - \beta_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{1-\beta_2} h(s, u(s, x))dW_s. \end{aligned} \quad (1.5)$$

Since

$$\begin{aligned} \int_0^t (t-s)^{-\beta_2} \int_0^s h(a, u(a, x))dW_a ds &= \int_0^t \int_a^t (t-s)^{-\beta_2} ds h(a, u(a, x))dW_a \\ &= \frac{1}{1 - \beta_2} \int_0^t (t-a)^{1-\beta_2} h(a, u(a, x))dW_a, \end{aligned}$$

by the definition of Caputo derivative (1.2) we have

$$\begin{aligned} \partial_t^{\beta_2} \int_0^t h(s, u(s, x))dW_s &= \frac{1}{\Gamma(1 - \beta_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\beta_2} \int_0^s h(a, u(a, x))dW_a ds \\ &= \frac{1}{\Gamma(2 - \beta_2)} \frac{\partial}{\partial t} \int_0^t (t-s)^{1-\beta_2} h(s, u(s, x))dW_s. \end{aligned}$$

Thus (1.5) can be rewritten as the following stochastic partial differential equation involving fractional time-derivative

$$\partial_t^{\beta_1} u = \operatorname{div} \vec{F} + \partial_t^{\beta_2} \int_0^t h(s, u(s, x))dW_s. \quad (1.6)$$

It is this type of stochastic equations and its natural extensions that will be studied in this paper.

Now let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ -fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, \mathbb{P})$ -null sets. We assume that on Ω we are given an independent family of one-dimensional Wiener processes W_t^1, W_t^2, \dots relative to the filtration $\{\mathcal{F}_t, t \geq 0\}$.

Motivated by (1.6), in this paper we consider the following quasi-linear SPDEs of the non-divergence form type

$$\begin{aligned} \partial_t^\beta u &= (a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + f(u)) \\ &\quad + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t (\sigma^{ijk}u_{x^i x^j} + \mu^{ik}u_{x^i} + \nu^k u + g^k(u)) dW_s^k \end{aligned} \quad (1.7)$$

as well as of the divergence form type

$$\begin{aligned} \partial_t^\beta u &= (D_i(a^{ij}u_{x^j} + b^i u + f^i(u)) + cu + h(u)) \\ &\quad + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t (\sigma^{ijk}u_{x^i x^j} + \mu^{ik}u_{x^i} + \nu^k u + g^k(u)) dW_s^k, \end{aligned} \quad (1.8)$$

given for $\omega \in \Omega$, $t \geq 0$ and $x \in \mathbb{R}^d$, and study the L_2 -theory of the equations. The constants $\beta, \gamma \in (0, 1)$ are assumed to satisfy the condition

$$\gamma < \beta + 1/2. \quad (1.9)$$

The indices i and j go from 1 to the dimension d with the summation convention on i, j being enforced. The coefficients $a^{ij}, b^i, c, \sigma^{ijk}, \mu^{ik}, \nu^k$ are functions depending on (ω, t, x) and the functions f, f^i, h, g^k depend on (ω, t, x) and the unknown u . By considering infinitely many independent Brownian motions W_t^k we cover equations driven by measure-valued processes, for instance, driven by space-time white noise (see Section 3.3). It is worth mentioning that unlike the classical SPDE, we allow the second-order derivatives of the unknown solution u to appear in the stochastic part when $\gamma < 1/2$.

As for stochastic differential equations (SDEs), SPDE (1.7) should be interpreted by its integral form

$$\begin{aligned} &u(t, x) - u(0, x) \\ &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (a^{ij}(s, x)u_{x^i x^j}(s, x) + \cdots + f(s, u(s, x))) ds \\ &\quad + \frac{1}{\Gamma(1+\beta-\gamma)} \int_0^t (t-s)^{\beta-\gamma} (\sigma^{ijk}(s, x)u_{x^i x^j}(s, x) + \cdots + g^k(s, u(s, x))) dW_s^k. \end{aligned}$$

Similarly one can write down the integral version of SPDE (1.8) but in the distributional sense with respect to x variable.

We next explain the constraint (1.9). A special case of the SPDEs for both (1.7) and (1.8) is

$$\partial_t^\beta u(t, x) = \Delta u(t, x) + \partial_t^\gamma \int_0^t g(s, x) dW_s, \quad (1.10)$$

where W is a one-dimensional Brownian motion. For functions h_1 and h_2 on \mathbb{R}_+ , we define its convolution $h_1 * h_2$ by

$$h_1 * h_2(t) = \int_0^t h_1(t-s)h_2(s)ds.$$

Let

$$k_\beta(t) := \Gamma(\beta)^{-1}t^{\beta-1},$$

and define

$$I^\beta \varphi = k_\beta * \varphi := \int_0^t k_\beta(t-s)\varphi(s)ds.$$

One can easily check for any $\beta, \gamma \in (0, 1)$, $k_\beta * k_\gamma = k_{\beta+\gamma}$. So in particular, we have

$$k_\beta * k_{1-\beta}(t) \equiv 1. \quad (1.11)$$

Suppose $u(x, t)$ is a solution of (1.10). In view of the definition of Caputo derivative (1.1), equation (1.10) is understood by its integral version

$$k_{1-\beta} * (u(t, x) - u(0, x)) = \int_0^t \Delta u(s, x) ds + k_{1-\gamma} * \int_0^t g(s) dW_s.$$

By taking convolution with k_β on both sides, we get from (1.10) and (1.11)

$$\int_0^t (u(s, x) - u(0, x)) ds = \left(I^\beta * \int_0^\cdot \Delta u(s, x) ds \right) (t) + \left(I^{\beta+1-\gamma} * \int_0^\cdot g(s, x) dW_s \right) (t).$$

Since the first two terms are differentiable in t , the last term above should be differentiable in t . Recall that

$$I^a : C^b \rightarrow C^{a+b},$$

and $\int_0^t g(s) dW_s \in C^{1/2-\varepsilon}$ for any $\varepsilon > 0$. Thus we must have

$$\beta + 1 - \gamma > 1/2,$$

which is equivalent to (1.9).

The main results of this paper are Theorems 4.3 and 5.4, on the unique solvability of SPDEs (1.7) and (1.8), and L_2 -estimates of their solutions. For SPDE (1.7), we establish in Theorem 5.4 the unique solvability in the space $L_2(\Omega \times [0, T], H_2^\sigma)$ for any $\sigma \in \mathbb{R}$ under appropriate differentiability assumption on x -variable of the coefficients. On the other hand, the unique solvability of SPDE (1.8) in the space $L_2(\Omega \times [0, T], H_2^1)$ is obtained in Theorem 4.3 under the merely measurability condition of the coefficients a^{ij} .

The rest of the paper is organized as follows. In Section 2 we present some preliminary results on the fractional derivatives and in Section 3 we introduce stochastic Banach spaces and few key estimates. Our main results for (1.8) and (1.7) are presented and proved in Section 4 and Subsection 5.1, respectively. Subsection 5.2 contains an application to an equation driven by space-time white noise.

We close this section with some notation. As usual, \mathbb{R}^d stands for the Euclidean space of points $x = (x^1, \dots, x^d)$. For $i = 1, \dots, d$, multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \dots\}$, and functions $u(x)$ we set

$$u_{x^i} = \partial u / \partial x^i = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

We also use the notation D_x^m for a partial derivative of order m with respect to x . By $C_0^\infty(\mathbb{R}^d)$ we denote the collection of smooth functions having compact support in \mathbb{R}^d . For $p \geq 1$, let

$$L_p = L_p(\mathbb{R}^d) := \{u : \mathbb{R}^d \rightarrow \mathbb{R}, \|u\|_{L_p}^p := \int_{\mathbb{R}^d} |u(x)|^p dx < \infty\}$$

and we use the notation $(f, g)_{L_2} := \int_{\mathbb{R}^d} f(x)g(x)dx$. We denote

$$\mathcal{F}(g)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} g(x) dx \quad \text{and} \quad \mathcal{F}^{-1}(f)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x) dx$$

the Fourier transform of g in \mathbb{R}^d and the inverse Fourier transform of f in \mathbb{R}^d , respectively.

If we write $N = N(a, b, \dots)$, this means that the constant N depends only on a, b, \dots . Throughout this paper, for functions depending on (ω, t, x) , usually the argument $\omega \in \Omega$ will be omitted.

2. Preliminary results

First we introduce a few elementary facts on the fractional derivatives. The reader can find further details in [2] and references therein. Recall that $\beta \in (0, 1)$ and

$$k_\beta(t) := t^{\beta-1} \Gamma(\beta)^{-1}, \quad t > 0.$$

Let $T > 0$. If f is absolutely continuous on $[0, T]$ with $f(0) = 0$ then

$$\frac{d}{dt}(k_\beta * f) = k_\beta * \frac{d}{dt}f, \quad t \in [0, T]. \quad (2.1)$$

For functions $\varphi \in L_1([0, T])$, the Riemann-Liouville fractional integral of the order $\beta \in (0, 1)$ is defined by

$$I^\beta \varphi(t) = k_\beta * \varphi(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \varphi(s) ds.$$

Note that by Jensen's inequality

$$|I^\beta \varphi(t)|^p \leq \frac{1}{(\Gamma(\beta))^p} \left(\frac{t^\beta}{\beta} \right)^{p-1} \int_0^t (t-s)^{\beta-1} |\varphi(s)|^p ds.$$

Thus it follows that for any $p \in [1, \infty]$,

$$\|I^\beta \varphi\|_{L_p([0, T])} \leq N(T, \beta) \|\varphi\|_{L_p([0, T])}. \quad (2.2)$$

Consequently, if $\varphi_n \rightarrow \varphi$ in $L_p([0, T])$ then $I^\beta \varphi_n$ also converges to $I^\beta \varphi$ in $L_p([0, T])$. Also one can prove that if $f_n(\omega, t)$ converges in probability uniformly in $[0, T]$ then so does $I^\beta f_n$.

If $I^{1-\beta} \varphi$ is absolutely continuous, then Riemann-Liouville derivative of order β is defined by

$$D_t^\beta \varphi(t) = \frac{d}{dt} (I^{1-\beta} \varphi)(t). \quad (2.3)$$

If φ is continuous and $I^{1-\beta} \varphi$ is absolutely continuous, the generalized functional derivative (or the Caputo derivative) of order β is given by

$$\partial_t^\beta \varphi(t) = D_t^\beta (\varphi - \varphi(0)) = D_t^\beta \varphi(t) - \frac{\varphi(0)}{t^\beta \Gamma(1-\beta)}. \quad (2.4)$$

It is easy to check for any $\varphi \in L_1([0, T])$,

$$D_t^\beta I^\beta \varphi = \varphi. \quad (2.5)$$

Furthermore, if φ is absolutely continuous on $[0, T]$ then by (2.1)

$$\partial_t^\beta \varphi = I^{1-\beta} \frac{d}{dt} \varphi. \quad (2.6)$$

Thus by (2.5)

$$D_t^{1-\beta} \partial_t^\beta \varphi = \frac{d}{dt} \varphi, \quad \text{a.e.} \quad (2.7)$$

Denote by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, \quad z \in \mathbb{C}$$

the Mittag-Leffler function. We will also use the generalized Mittag-Leffler function

$$E_{\beta, \gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}.$$

We assume $\beta, \gamma \in (0, 1)$. It is well known (see e.g. (12) in [1, Theorem 1.3-4]) that, when $-1 < \gamma - \beta < 1$, $E_{\beta, \gamma}(t)$ is bounded on $(-\infty, 0]$ and

$$\lim_{\mathbb{R} \ni t \rightarrow \infty} t E_{\beta, \gamma}(-t) = \frac{1}{\Gamma(\gamma - \beta)}. \quad (2.8)$$

Furthermore, for any constant λ ,

$$\varphi(t) := E_\beta(\lambda t^\beta) \quad (2.9)$$

is a solution of the equation

$$\partial_t^\beta \varphi = \lambda \varphi, \quad t > 0$$

with the initial condition $\varphi(0) = 1$.

The following is a classical result. We provide a proof for the readers' convenience.

Lemma 2.1. *Let $a \in (0, 1)$ and $b \geq 0$, then*

$$D_t^a E_\beta(-bt^\beta) = t^{-a} E_{\beta, 1-a}(-bt^\beta), \quad (2.10)$$

$$I^a E_\beta(-bt^\beta) = t^a E_{\beta, 1+a}(-bt^\beta). \quad (2.11)$$

Proof. We first prove (2.10). One can easily check (see e.g. [2, (5.1.2)])

$$D_t^a t^{b-1} = \frac{\Gamma(b)}{\Gamma(b-a)} t^{b-a-1}, \quad a, b > 0. \quad (2.12)$$

Thus,

$$\begin{aligned} D_t^a E_\beta(-bt^\beta) &= D_t^a \left(\sum_{k=0}^{\infty} \frac{(-1)^k b^{2k} t^{\beta k}}{\Gamma(\beta k + 1)} \right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k b^{2k}}{\Gamma(\beta k + 1)} \frac{\Gamma(\beta k + 1)}{\Gamma(\beta k + 1 - a)} t^{\beta k - a} \\ &= t^{-a} \sum_{k=0}^{\infty} \frac{(-bt^\beta)^k}{\Gamma(\beta k + 1 - a)} \\ &= t^{-a} E_{\beta, 1-a}(-bt^\beta). \end{aligned}$$

To prove (2.11), in place of (2.12), it is enough to use

$$I^a t^{b-1} = \frac{\Gamma(b)}{\Gamma(b+a)} t^{b+a-1}, \quad a, b > 0.$$

The lemma is proved. \square

Define

$$p(t, x) = \mathcal{F}^{-1}(E_\beta(-|\xi|^2 t^\beta)), \quad q(t, x) = D_t^{1-\beta} p.$$

Actually (2.8) shows that if $d > 1$ then $E_\beta(-|\xi|^2 t^\beta) \notin L_1(\mathbb{R}^d)$ for fixed $t > 0$. Thus we understand $p(t, x)$ as the inverse transform of a radial function in the sense of improper integral, or we can define $p(t, x)$ first as in [2, Section 5.2.2] so that $p(t, \cdot) \in L_1(\mathbb{R}^d)$ and $\mathcal{F}(p(t, x))(\xi) = E_\beta(-|\xi|^2 t^\beta)$.

Since, for $x \neq 0$, $p(t, x) \rightarrow 0$ as $t \downarrow 0$ (see [2]), the Riemann-Liouville derivative of $p(\cdot, x)$ coincides with the Caputo derivative of $p(\cdot, x)$ for every $x \neq 0$.

Here is a list of other useful properties of p and q .

Lemma 2.2. (i) *For $(t, x) \in (0, T] \times \mathbb{R}^d \setminus \{0\}$, we have $\partial_t^\beta p = D_t^\beta p = \Delta p$.*

(ii) *For each $x \neq 0$ and $m \leq 3$,*

$$\lim_{t \rightarrow 0+} D_x^m p(t, x) = \lim_{t \rightarrow 0+} D_x^m q(t, x) = 0. \quad (2.13)$$

(iii) *$D_x^m p(t, \cdot)$ is integrable for each $t > 0$ if $m \leq 1$.*

(iv) *For each $t > 0$ and $m \leq 3$, $D_x^m q(t, \cdot)$ is integrable in \mathbb{R}^d uniformly on $[\varepsilon, T]$ for any $\varepsilon > 0$.*

(v) *For each $x \neq 0$, $p(\cdot, x)$ is absolutely continuous and $\frac{\partial}{\partial t} p(t, x) \rightarrow 0$ as $t \downarrow 0$. Moreover, $\frac{\partial}{\partial t} p(t, \cdot)$ is integrable in \mathbb{R}^d uniformly on $[\varepsilon, T]$ for any $\varepsilon > 0$.*

(vi) *For any compact set $K \subset \mathbb{R}^d \setminus \{0\}$ and $m \leq 3$, the functions $p, q, \frac{\partial}{\partial t} p(t, \cdot), D^m p$ and $D^m q$ are continuous and bounded on $[0, T] \times K$.*

Proof. (i) is a consequence of (2.9). See (5.2.44) and (A.19) of [2] for (v). Others can be found in Propositions 5.1 and 5.2 of [2]. In particular, (vi) is a consequence of (5.2.6) and (5.2.13) of [2]. \square

We need the following integration by parts formula.

Lemma 2.3. *Suppose that f is absolutely continuous on $[0, T]$ with $f(T) = 0$ and g is continuous and $I^{1-\beta}g$ is absolutely continuous on $[0, T]$. Then*

$$\int_0^T f(t) D_t^\beta g(t) dt = \int_0^T G(t) D_t^\beta F(t) dt,$$

where $F(t) = f(T - t)$ and $G(t) = g(T - t)$.

Proof. Note that

$$\int_0^T |f'(t)| \left(\int_0^t \theta^{-\beta} |g(t - \theta)| d\theta \right) dt \leq \|g\|_{L^\infty([0, T])} \int_0^T |f'(t)| dt \int_0^T \theta^{-\beta} d\theta < \infty.$$

Thus, using $f(T) = 0$, the integration by part and the Fubini's Theorem, we get

$$\begin{aligned} \int_0^T f(t) D_t^\beta g(t) dt &= \int_0^T f(t) \frac{d}{dt} (I^{1-\beta} g)(t) dt \\ &= f(t) (I^{1-\beta} g)(t) \Big|_0^T - \int_0^T f'(t) (I^{1-\beta} g)(t) dt \\ &= -\frac{1}{\Gamma(1-\beta)} \int_0^T f'(t) \left(\int_0^t \theta^{-\beta} g(t - \theta) d\theta \right) dt \\ &= -\frac{1}{\Gamma(1-\beta)} \int_0^T \theta^{-\beta} \left(\int_\theta^T f'(t) g(t - \theta) dt \right) d\theta. \end{aligned} \quad (2.14)$$

As $F(0) = f(T) = 0$ and f is absolutely continuous, by (2.4) and (2.6) we have $D_t^\beta F = \partial_t^\beta F = (I^{1-\beta} F')$. So by the integration by part and the Fubini's Theorem

$$\begin{aligned} \int_0^T G(t) D_t^\beta F(t) dt &= \int_0^T G(t) (I^{1-\beta} F')(t) dt \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^T G(t) \left(\int_0^t \theta^{-\beta} F'(t - \theta) d\theta \right) dt \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^T \theta^{-\beta} \left(\int_\theta^T G(t) F'(t - \theta) dt \right) d\theta \\ &= -\frac{1}{\Gamma(1-\beta)} \int_0^T \theta^{-\beta} \left(\int_\theta^T f'(T - t + \theta) g(T - t) dt \right) d\theta \\ &= -\frac{1}{\Gamma(1-\beta)} \int_0^T \theta^{-\beta} \left(\int_\theta^T f'(s) g(s - \theta) ds \right) d\theta, \end{aligned}$$

which is $\int_0^T f(s) D_t^\beta g(s) ds$ by (2.14). This proves the lemma. \square

Lemma 2.4. *For each $(t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}$,*

$$\Delta q(t, x) = \frac{\partial}{\partial t} p(t, x). \quad (2.15)$$

Proof. Since both are continuous it is enough to prove that the equality holds for almost all (t, x) . Let $\phi(t)$ and $\psi(x)$ be smooth functions with compact support in $(0, T)$ and $\mathbb{R}^d \setminus \{0\}$ respectively. Define $\Phi(s) := \phi(T-s)$. Since $\phi(t)$ is smooth function with compact support in $(0, T)$, using Lemma 2.2(i), Lemma 2.3 and (2.7), for every $x \in \mathbb{R}^d$

$$\begin{aligned} \int_0^T D_t^{1-\beta} \Phi(s) \Delta p(T-s, x) ds &= \int_0^T D_t^{1-\beta} \Phi(s) \left(\partial_t^\beta p(\cdot, x) \right) (T-s) ds \\ &= \int_0^T \phi(s) \left(D_t^{1-\beta} \partial_t^\beta p(\cdot, x) \right) (s) ds \\ &= \int_0^T \phi(s) \frac{\partial}{\partial s} p(s, x) ds. \end{aligned} \quad (2.16)$$

Recall that $D_x^2 q$, $D_x^2 p$ and $\frac{\partial}{\partial s} p$ are locally integrable in $\mathbb{R}^d \setminus \{0\}$ uniformly on the support of $\phi(t)$. By the integration by parts in x , Lemma 2.2(vi), Lemma 2.3, (2.16) and the Fubini's Theorem,

$$\begin{aligned} \int_{\mathbb{R}^d} \int_0^T \phi(s) \psi(x) \Delta q(s, x) ds dx &= \int_0^T \int_{\mathbb{R}^d} \phi(s) \Delta \psi(x) q(s, x) dx ds \\ &= \int_{\mathbb{R}^d} \Delta \psi(x) \left(\int_0^T \phi(s) D_t^{1-\beta} p(s, x) ds \right) dx \\ &= \int_{\mathbb{R}^d} \Delta \psi(x) \left(\int_0^T D_t^{1-\beta} \Phi(s) p(T-s, x) ds \right) dx \\ &= \int_{\mathbb{R}^d} \psi(x) \left(\int_0^T D_t^{1-\beta} \Phi(s) \Delta p(T-s, x) ds \right) dx \\ &= \int_{\mathbb{R}^d} \int_0^T \phi(s) \psi(x) \frac{\partial}{\partial s} p(s, x) ds dx. \end{aligned}$$

Since $\phi(t)$ and $\psi(x)$ are arbitrary, the lemma is proved. \square

3. Key Estimates

In this section, we first define a stochastic Banach space and establish key lemmas. Then we study the L_2 -theory of a model equation for SPDEs with fractional time-derivatives.

For $n = 0, 1, 2, \dots$, define the Banach spaces

$$H_2^n := H_2^n(\mathbb{R}^d) = \{u : u, D_x u, \dots, D_x^n u \in L_2\}.$$

In general, for $\sigma \in \mathbb{R}$ define the space $H_2^\sigma = H_2^\sigma(\mathbb{R}^d) = (1 - \Delta)^{-\sigma/2} L_2$ as the set of all distributions u on \mathbb{R}^d such that $(1 - \Delta)^{\sigma/2} u \in L_2$. For $u \in H_2^\sigma$, we define

$$\|u\|_{H_2^\sigma} := \|(1 - \Delta)^{\sigma/2} u\|_{L_2} := \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\sigma/2} \mathcal{F}(u)(\xi)]\|_{L_2}. \quad (3.1)$$

Similarly for ℓ_2 -valued $g = (g^1, g^2, \dots)$,

$$\|g\|_{H_2^\sigma(\ell_2)} := \|(1 - \Delta)^{\sigma/2} g\|_{\ell_2} := \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\sigma/2} \mathcal{F}(g)(\xi)]\|_{\ell_2}. \quad (3.2)$$

Let \mathcal{P} be the predictable σ -field and $\mathcal{P}^{d\mathbb{P} \times dt}$ be the completion of \mathcal{P} with respect to $d\mathbb{P} \times dt$. For each $\sigma \in \mathbb{R}$, define the Banach space

$$\mathbb{H}_2^\sigma(T) := L_2(\Omega \times [0, T], \mathcal{P}, H_2^\sigma).$$

That is, $u \in \mathbb{H}_2^\sigma(T)$ if u is an H_2^σ -valued $\mathcal{P}^{d\mathbb{P} \times dt}$ -measurable process defined on $\Omega \times [0, T]$ so that

$$\|u\|_{\mathbb{H}_2^\sigma(T)} := \left(\mathbb{E} \int_0^T \|u(t, \cdot)\|_{H_2^\sigma}^2 dt \right)^{1/2} < \infty.$$

For an ℓ_2 -valued process $g = (g^1, g^2, \dots)$, we write $g \in \mathbb{H}_2^\sigma(T, \ell_2)$ if $g^k \in \mathbb{H}_2^\sigma(T)$ for every $k \geq 1$ and

$$\|g\|_{\mathbb{H}_2^\sigma(T, \ell_2)} := \left(\mathbb{E} \int_0^T \|g\|_{H_2^\sigma(\ell_2)}^2 dt \right)^{1/2} < \infty.$$

Denote $\mathbb{L}_2(T) = \mathbb{H}_2^0(T)$ and $\mathbb{L}_2(T, \ell_2) = \mathbb{H}_2^0(T, \ell_2)$. Write $g \in \mathbb{H}_0^\infty(T, \ell_2)$ if $g^k = 0$ for all sufficiently large k , and each g^k is of the type

$$g^k = \sum_{i=1}^n I_{(\tau_{i-1}, \tau_i]}(t) g^{ik}(x)$$

where τ_i are bounded stopping times with respect to \mathcal{F}_t and $g^{ik} \in C_0^\infty(\mathbb{R}^d)$. It is known ([5, Theorem 3.10]) that $\mathbb{H}_0^\infty(T, \ell_2)$ is dense in $\mathbb{H}_2^\sigma(T, \ell_2)$ for any σ .

Finally we use U_2^σ to denote the family of $H_2^\sigma(\mathbb{R}^d)$ -valued \mathcal{F}_0 -measurable random variables u_0 having

$$\|u_0\|_{U_2^\sigma} := \left(\mathbb{E} \|u_0\|_{H_2^\sigma}^2 \right)^{1/2} < \infty.$$

Lemma 3.1. *Suppose $a > 0$. (i) Let $h = (h^1, h^2, \dots) \in L_2(\Omega \times [0, T], \mathcal{P}, \ell_2)$. Then the equality*

$$I^a \left(\sum_{k=1}^{\infty} \int_0^\cdot h^k(s) dW_s^k \right)(t) = \sum_{k=1}^{\infty} \left(I^a \int_0^\cdot h^k(s) dW_s^k \right)(t)$$

holds for all $t \leq T$ (a.s.) and also in $L_2(\Omega \times [0, T])$.

(ii) *Suppose $h_n = (h_n^1, h_n^2, \dots)$ converges to $h = (h^1, h^2, \dots)$ in $L_2(\Omega \times [0, T], \mathcal{P}, \ell_2)$ as $n \rightarrow \infty$. Then, as $n \rightarrow \infty$,*

$$\sum_{k=1}^{\infty} I^a \int_0^\cdot h_n^k dW_s^k \quad \text{converges to} \quad \sum_{k=1}^{\infty} I^a \int_0^\cdot h^k(s) dW_s^k \quad (3.3)$$

in probability uniformly on $[0, T]$.

Proof. Since the series $\sum_{k=1}^{\infty} \int_0^t h^k(s) dW_s^k$ converges in $L_2(\Omega \times [0, T])$, by (2.2) we have

$$\begin{aligned} I^a \left(\sum_{k=1}^{\infty} \int_0^\cdot h^k(s) dW_s^k \right)(t) &= \lim_{n \rightarrow \infty} I^a \left(\sum_{k=1}^n \int_0^\cdot h^k(s) dW_s^k \right)(t) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n I^a \left(\int_0^\cdot h^k(s) dW_s^k \right)(t) \quad \text{in } L_2(\Omega \times [0, T]). \end{aligned}$$

Thus, the series $\sum_{k=1}^{\infty} I^a \int_0^t h^k(s) dW_s^k$ converges in $L_2(\Omega \times [0, T])$ and

$$I^a \sum_{k=1}^{\infty} \int_0^t h^k(s) dW_s^k = \sum_{k=1}^{\infty} I^a \int_0^t h^k(s) dW_s^k \quad (3.4)$$

in $L_2(\Omega \times [0, T])$, and thus the equality holds (a.e.). Also by Burkholder-Davis-Gundy inequality

$$\mathbb{E} \left[\sup_{t \leq T} \left| \sum_{k=n}^m I^a \int_0^t h^k(s) dW_s^k \right|^2 \right]$$

$$\leq N \mathbb{E} \left[\sup_{t \leq T} \left| \sum_{k=n}^m \int_0^t h^k(s) dW_s^k \right|^2 \right] \leq N \sum_{k=n}^m \mathbb{E} \left[\int_0^T |h^k(s)|^2 ds \right] \rightarrow 0 \quad (3.5)$$

as $n, m \rightarrow \infty$. Therefore the series $\sum_{k=1}^{\infty} I^a \int_0^t h^k(s) dW_s^k$ converges in probability uniformly on $[0, T]$. It follows that both sides of (3.4) are continuous, and therefore the equality above holds for all $t \leq T$ (a.s.). \square

Remark 3.2. Let $\sigma \in \mathbb{R}$. Suppose $g_n \rightarrow g$ in $\mathbb{H}_2^\sigma(T, \ell_2)$ as $n \rightarrow \infty$, and $\phi \in C_0^\infty(\mathbb{R}^d)$. Then $(g_n(t, \cdot), \phi)_{L_2} \rightarrow (g(t, \cdot), \phi)_{L_2}$ in $L_2(\Omega \times [0, T], \mathcal{P}, \ell_2)$, and therefore Lemma 3.1(ii) holds with $h_n(t) := (g_n(t, \cdot), \phi)_{L_2}$ and $h(t) := (g(t, \cdot), \phi)_{L_2}$.

Lemma 3.3. *Let $\alpha > 1/2$ and $g \in \mathbb{H}_0^\infty(T, \ell_2)$. Then $I^\alpha \sum_{k=1}^{\infty} \int_0^\cdot g^k(s) dW_s^k$ is differentiable in t and (a.s.) for all $t \leq T$*

$$\frac{\partial}{\partial t} (I^\alpha \sum_{k=1}^{\infty} \int_0^\cdot g^k(s) dW_s^k)(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t (t-s)^{\alpha-1} g^k(s) dW_s^k.$$

Proof. We integrate the right hand side and then use the stochastic Fubini's theorem (see e.g. [6]) to get

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t \int_0^s (s-r)^{\alpha-1} g^k(r) dW_r^k ds &= \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t \int_r^t (s-r)^{\alpha-1} ds g^k(r) dW_r^k \\ &= \frac{1}{\alpha \Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t (t-r)^\alpha g^k(r) dW_r^k. \end{aligned}$$

Similarly, by Lemma 3.1(i)

$$\begin{aligned} I^\alpha \sum_{k=1}^{\infty} \left(\int_0^\cdot g^k(s) dW_s^k \right)(t) &= \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t (t-s)^{\alpha-1} \left(\int_0^s g^k(r) dW_r^k \right) ds \\ &= \frac{1}{\alpha \Gamma(\alpha)} \sum_{k=1}^{\infty} \int_0^t (t-r)^\alpha g^k(r) dW_r^k. \end{aligned}$$

The lemma is proved. \square

Lemma 3.3 can be easily extended for any $g \in \mathbb{L}_2(T, \ell_2)$.

For the remainder of this paper, we assume that (1.9) holds. For $a \in \mathbb{R}$, denote $a_+ = \max\{a, 0\}$. Define

$$\gamma_0 := \frac{(2\gamma - 1)_+}{\beta} < 2. \quad (3.6)$$

Note that since $\gamma < \beta + 1/2$ we have

$$\gamma_0 < 2, \quad \text{and} \quad \gamma_0 = 0 \quad \text{if} \quad \gamma \leq 1/2. \quad (3.7)$$

Definition 3.4. We write $u \in \mathcal{H}_2^{\sigma+2}(T)$ if $u \in \mathbb{H}_2^{\sigma+2}(T)$, $u(0) \in U_2^{\sigma+1}$, and for some $f \in \mathbb{H}_2^\sigma(T)$ and $g \in \mathbb{H}_2^{\sigma+\gamma_0}(T, \ell_2)$ it holds that

$$\partial_t^\beta u(t, x) = f(t, x) + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t g^k(s, x) dW_s^k \quad (3.8)$$

in the sense of distributions. That is, for any $\phi \in C_0^\infty(\mathbb{R}^d)$ the equality

$$(I^{1-\beta}(u - u(0))(t), \phi)_{L_2} = \int_0^t (f(s, \cdot), \phi)_{L_2} ds + \sum_{k=1}^{\infty} (I^{1-\gamma} \int_0^t (g^k(s, \cdot), \phi)_{L_2} dW_s^k)(t) \quad (3.9)$$

holds for all $t \leq T$ (a.s.). In this case we write

$$f = \mathbb{D}u, \quad g = \mathbb{S}u$$

and define

$$\|u\|_{\mathcal{H}_2^{\sigma+2}(T)} = \|u(0)\|_{U_2^{\sigma+1}} + \|u\|_{\mathbb{H}_2^{\sigma+2}(T)} + \|\mathbb{D}u\|_{\mathbb{H}_2^\sigma(T)} + \|\mathbb{S}u\|_{\mathbb{H}_2^{\sigma+\gamma_0}(T, \ell_2)}. \quad (3.10)$$

Finally define

$$\mathcal{H}_{2,0}^{\sigma+2}(T) = \mathcal{H}_2^{\sigma+2}(T) \cap \{u : u(0) = 0\}. \quad (3.11)$$

Remark 3.5. By (1.11), (2.1) and Lemma 3.3, (3.9) is equivalent to

$$\begin{aligned} (u(t, \cdot) - u(0, \cdot), \phi)_{L_2} &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (f(s, \cdot), \phi)_{L_2} ds \\ &\quad + \frac{1}{\Gamma(1+\beta-\gamma)} \sum_{k=1}^{\infty} \int_0^t (t-s)^{\beta-\gamma} (g^k(s, \cdot), \phi)_{L_2} dW_s^k. \end{aligned}$$

Lemma 3.6. (i) $\mathcal{H}_2^{\sigma+2}(T)$ is a Banach space.

- (ii) Let $u \in \mathcal{H}_2^{\sigma+2}(T)$. Then u is a continuous H_2^σ -valued process.
 (iii) Assume that $u \in \mathcal{H}_2^2(T)$ and (3.8) holds. Then (a.s.)

$$(k_{1-\beta} * \|u - u(0) - v\|_{L_2}^2)(t) \leq 2 \int_0^t (f(s, \cdot), u(s, \cdot) - u(0, \cdot) - v(s, \cdot))_{L_2} ds \quad \text{for } t \leq T, \quad (3.12)$$

where

$$v(t, x) = \Gamma(1+\beta-\gamma)^{-1} \sum_{k=1}^{\infty} \int_0^t (t-s)^{\beta-\gamma} g^k(s, x) dW_s^k. \quad (3.13)$$

Proof. (i) We only need to prove the completeness of the space. Let u_n , $n = 1, 2, \dots$, be a Cauchy sequence in $\mathcal{H}_2^{\sigma+2}(T)$ with

$$f_n = \mathbb{D}u_n, \quad g_n = \mathbb{S}u_n.$$

Then there exist u, f, g, u_0 so that $u_n, f_n, g_n, u_n(0)$ converge to u, f, g, u_0 respectively in their corresponding spaces. To prove $u_n \rightarrow u$ in $\mathcal{H}_2^{\sigma+2}(T)$, it suffices to show (3.9) holds for all $t \leq T$ (a.s.). Since the series $\sum_{k=1}^{\infty} \int_0^t (g_n^k(s), \phi) dW_s^k$ converges in probability uniformly on $[0, T]$, so does $(I^{1-\gamma} \sum_{k=1}^{\infty} \int_0^t (g_n^k(s), \phi) dW_s^k)(t)$. By Remark 3.2, considering the limit of

$$(I^{1-\beta}(u_n - u_n(0)))(t), \phi)_{L_2} = \int_0^t (f_n(s, \cdot), \phi)_{L_2} ds + \sum_{k=1}^{\infty} (I^{1-\gamma} \int_0^t (g_n^k(s, \cdot), \phi)_{L_2} dW_s^k)(t)$$

for $t \leq T$, we get (3.9) for all $t \leq T$ (a.s.) since both sides of (3.9) are continuous in t .

(ii) We only prove the case $\sigma = 0$. The general case is covered by applying $(1 - \Delta)^{\sigma/2}$ to (3.8). Denote $f = \mathbb{D}u$ and $g = \mathbb{S}u$. Notice that as an $L_2(\mathbb{R}^d)$ -valued process, $u(t) - u(0)$ satisfies

$$k_{1-\beta} * (u(\cdot, x) - u(0, x))(t) = \int_0^t f(s, x) ds + (k_{1-\gamma} * (\sum_{k=1}^{\infty} \int_0^t g^k(s, x) dW_s^k))(t) \quad \text{all } t \leq T \text{ (a.s.)}.$$

Taking the convolution with k_β and using

$$\begin{aligned} \frac{\partial}{\partial t}(k_\beta * \int_0^\cdot f(s, x) ds)(t) &= (k_\beta * f)(t, x), \\ \frac{\partial}{\partial t}(k_\beta * k_{1-\gamma} * (\sum_{k=1}^\infty \int_0^\cdot g^k(s, x) dw_s))(t) &= \sum_{k=1}^\infty \frac{1}{\Gamma(1+\beta-\gamma)} \int_0^t (t-s)^{\beta-\gamma} g^k(s, x) dW_s^k, \end{aligned}$$

where the second equality is from Lemma 3.3, we get for all $t \leq T$ (a.s.)

$$u(t, x) - u(0, x) = (k_\beta * f)(t, x) + \sum_{k=1}^\infty \frac{1}{\Gamma(1+\beta-\gamma)} \int_0^t (t-s)^{\beta-\gamma} g^k(s, x) dW_s^k. \quad (3.14)$$

Hence the claim follows.

(iii) Denote $w(t, x) = u(t, x) - u(0, x) - v(t, x)$. Then we have $\partial_t^\beta w(t, x) = f(t, x)$. Now we use the fact (see e.g. [14, Lemma 2.1]) that if κ is a positive decreasing function on $[0, T]$ then

$$\kappa * \|w\|_{L_2}^2(t) \leq 2 \int_0^t \left(\frac{\partial}{\partial s} (\kappa * w)(s, \cdot), w(s, \cdot) \right)_{L_2} ds.$$

We take (see [14]) a sequence of such functions $\kappa_n \in H_1^1([0, T])$ so that $\kappa_n \rightarrow k_{1-\beta}$ in $L_1([0, T])$ and $\frac{\partial}{\partial t}(\kappa_n * w) \rightarrow \frac{\partial}{\partial t}(k_{1-\beta} * w)$ in $L_2([0, T], L_2)$. Finally for (3.12) it is enough to note that

$$\kappa_n * \|w\|_{L_2}^2 \rightarrow k_{1-\beta} * \|w\|_{L_2}^2 \quad \text{in } L_1([0, T])$$

and both sides of (3.12) are continuous in t . The lemma is proved. \square

Recall that for any σ ,

$$\|u\|_{\mathbb{H}_2^\sigma(t)}^2 := \mathbb{E} \int_0^t \|u(s)\|_{H_2^\sigma}^2 ds.$$

Proposition 3.7. Let $u \in \mathcal{H}_2^{\sigma+2}(T)$. Then for any $t \leq T$,

$$(k_{1-\beta} * \mathbb{E}\|u\|_{H_2^\sigma}^2)(t) \leq N(\mathbb{E}\|u(0)\|_{H_2^\sigma}^2 + \|\mathbb{D}u\|_{\mathbb{H}_2^\sigma(t)}^2 + \|\mathbb{S}u\|_{\mathbb{H}_2^\sigma(t, \ell_2)}^2 + \|u\|_{\mathbb{H}_2^\sigma(t)}^2) \leq N\|u\|_{\mathcal{H}_2^{\sigma+2}(t)}^2, \quad (3.15)$$

where N depends only on T, β and γ . In particular, for any $t \leq T$,

$$\|u\|_{\mathbb{H}_2^\sigma(t)}^2 \leq N \int_0^t (t-s)^{-1+\beta} \|u\|_{\mathcal{H}_2^{\sigma+2}(s)}^2 ds. \quad (3.16)$$

Proof. We only consider the case $\sigma = 0$. In general, one can consider $\Delta^{\sigma/2}u$ in place of u . Denote v as (3.13) in Lemma 3.6. Then by (3.12),

$$(k_{1-\beta} * \mathbb{E}\|u\|_{L_2}^2)(t) \leq 2(k_{1-\beta} * \mathbb{E}\|u(0) + v\|_{L_2}^2)(t) + 2\mathbb{E} \int_0^t (f(s, \cdot), u(s, \cdot) - u(0, \cdot) - v(s, \cdot))_{L_2} ds.$$

Note that by Fubini's theorem and Davis's inequality

$$\begin{aligned} \mathbb{E} \int_0^t \|v(s)\|_{L_2}^2 ds &= \int_0^t \mathbb{E} \|v(s)\|_{L_2}^2 ds \\ &\leq N \mathbb{E} \int_0^t \int_0^s (s-r)^{2(\beta-\gamma)} \|g(r, \cdot)\|_{L_2(\ell_2)}^2 dr ds \leq N \|g\|_{\mathbb{L}_2(t, \ell_2)}^2. \end{aligned} \quad (3.17)$$

For the last inequality we use the fact $2(\beta-\gamma) > -1$. Therefore, by young's inequality

$$\mathbb{E} \int_0^t (f(s, \cdot), u(s, \cdot) - u(0, \cdot) - v(s, \cdot))_{L_2} ds \leq N(\mathbb{E}\|u(0)\|_{L_2}^2 + \|f\|_{\mathbb{L}_2(t)}^2 + \|g\|_{\mathbb{L}_2(t, \ell_2)}^2 + \|u\|_{\mathbb{L}_2(t)}^2).$$

Also,

$$(k_{1-\beta} * \mathbb{E}\|u(0) + v\|_{L_2}^2)(t) \leq N\mathbb{E}\|u(0)\|_{L_2}^2 + Nk_{1-\beta} * \mathbb{E}\|v\|_{L_2}^2(t),$$

$$k_{1-\beta} * \mathbb{E}\|v\|_{L_2}^2(t) \leq N \int_0^t (t-s)^{-\beta} \int_0^s (s-r)^{2(\beta-\gamma)} \mathbb{E}\|g(r, \cdot)\|_{L_2(\ell_2)}^2 dr ds \leq N\|g\|_{\mathbb{L}_2(t, \ell_2)}^2.$$

This proves (3.15).

Note that the second equality in (3.15) just follows from the definition (3.10) of $\|u\|_{\mathcal{H}_2^{\sigma+2}(t)}^2$. Hence to prove (3.16) it is enough to consider a convolution with k_β and use (1.11), which implies

$$(k_\beta * k_{1-\beta} * \mathbb{E}\|u\|_{L_2}^2)(t) = \int_0^t \mathbb{E}\|u\|_{L_2}^2 ds = \|u\|_{\mathbb{L}_2(t)}^2.$$

Hence the theorem is now proved. \square

Define

$$P_{\beta, \gamma}(t, x) = \begin{cases} I^{\beta-\gamma} p(\cdot, x)(t) & \text{if } \beta \geq \gamma \\ \partial_t^{\gamma-\beta} p(\cdot, x)(t) & \text{if } \beta < \gamma. \end{cases}$$

Lemma 3.8. *Let $g \in \mathbb{H}_0^\infty(T, \ell_2)$ and u be defined by*

$$u(t, x) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} P_{\beta, \gamma}(t-s, x-y) g^k(s, y) dy dW_s^k.$$

Let $\sigma \leq 2 \wedge (\frac{1-2\gamma}{\beta} + 2)$ if $\gamma \neq 1/2$, and $\sigma < 2$ if $\gamma = 1/2$. Then it holds that

$$\mathbb{E} \int_0^T \|\Delta^{\sigma/2} u(t, \cdot)\|_{L_2}^2 dt \leq N\|g\|_{\mathbb{L}_2(T, \ell_2)}^2.$$

In general, for any $\gamma_1 \in \mathbb{R}$,

$$\mathbb{E} \int_0^T \|u(t, \cdot)\|_{H_2^{\gamma_1+\sigma}}^2 dt \leq N\|g\|_{\mathbb{H}_2^{\gamma_1}(T, \ell_2)}^2.$$

Proof. Let $a := \gamma - \beta < 1/2$. Recall that $E_{\alpha, \gamma}(t)$ is bounded on $(-\infty, 0]$. By the Fourier transform and Lemma 2.1, we have for any $\sigma \geq 0$

$$\begin{aligned} & \mathbb{E} \int_0^T \|\Delta^{\sigma/2} u(t, \cdot)\|_{L_2}^2 dt \\ & \leq N \sum_{k=1}^{\infty} \mathbb{E} \int_0^T \int_0^t \int_{\mathbb{R}^d} |\xi|^{2\sigma} (t-s)^{-2a} E_{\beta, 1-a}^2(-|\xi|^2(t-s)^\beta) |\hat{g}^k(s, \xi)|^2 d\xi ds dt \\ & \leq N\|g\|_{\mathbb{L}_2(T, \ell_2)}^2 \\ & \quad + N\mathbb{E} \int_0^T \int_s^T \int_{|\xi| \geq 1} |\xi|^{2\sigma} (t-s)^{-2a} E_{\beta, 1-a}^2(-|\xi|^2(t-s)^\beta) |\hat{g}(s, \xi)|_{\ell_2}^2 d\xi dt ds. \end{aligned}$$

By the substitution $r = |\xi|^{2/\beta}(t-s)$, the last term above is bounded by constant times of

$$\mathbb{E} \int_{|\xi| \geq 1} \int_0^T |\hat{g}(s, \xi)|_{\ell_2}^2 \int_0^{T|\xi|^{\frac{2}{\beta}}} |\xi|^{2(\sigma+(2a-1)/\beta)} r^{-2a} E_{\beta, 1-a}^2(-r^\beta) dr ds d\xi.$$

Let $\gamma > 1/2$. Then, since $E_{\beta,1-a}(-r^\beta)$ is bounded on $[0, \infty)$ and $E_{\beta,1-a}(-r^\beta) \leq Nr^{-\beta}$ if $r \geq 1$ (see (2.8)), we have

$$\begin{aligned} & \int_0^{T|\xi|^{\frac{2}{\beta}}} |\xi|^{2(\sigma+(2a-1)/\beta)} r^{-2a} E_{\beta,1-a}^2(-r^\beta) dr \\ & \leq \int_0^\infty r^{-2a} E_{\beta,1-a}^2(-r^\beta) dr \\ & \leq N \left(\int_0^1 r^{-2a} dr + \int_1^\infty r^{-2\gamma} dr \right) < \infty. \end{aligned}$$

If $\gamma = 1/2$, then since $|\xi| \geq 1$ and $\sigma < 2$,

$$\begin{aligned} & \int_0^{T|\xi|^{\frac{2}{\beta}}} |\xi|^{2(\sigma+(2a-1)/\beta)} r^{-2a} E_{\beta,1-a}^2(-r^\beta) dr \\ & = \int_0^{T|\xi|^{\frac{2}{\beta}}} |\xi|^{-2(2-\sigma)} r^{-2a} E_{\beta,1-a}^2(-r^\beta) dr \\ & \leq \int_0^1 r^{-2a} E_{\beta,1-a}^2(-r^\beta) dr + N|\xi|^{-2(2-\sigma)} \int_1^{T|\xi|^{\frac{2}{\beta}}} r^{-1} dr \\ & \leq N \int_0^1 r^{-2a} dr + N|\xi|^{-2(2-\sigma)} \ln |\xi| \leq N < \infty. \end{aligned}$$

The case $\gamma < 1/2$ is treated similarly using $\sigma \leq 2$. Indeed,

$$\begin{aligned} & \int_0^{T|\xi|^{\frac{2}{\beta}}} |\xi|^{2(\sigma+(2a-1)/\beta)} r^{-2a} E_{\beta,1-a}^2(-r^\beta) dr \\ & \leq \int_0^1 r^{-2a} E_{\beta,1-a}^2(-r^\beta) dr + N|\xi|^{2(\sigma+(2a-1)/\beta)} \int_1^{T|\xi|^{\frac{2}{\beta}}} r^{-2\beta} dr \\ & \leq N \int_0^1 r^{-2a} dr + N|\xi|^{2\sigma-4} \leq N < \infty. \end{aligned}$$

Therefore the lemma is proved. \square

Lemma 3.8 says that u (which is a solution of (3.18) below) is smoother than g by order $2 \wedge ((1 - 2\gamma)\beta^{-1} + 2)$ if $\gamma \neq 1/2$ and $2 - \varepsilon$ if $\gamma = 1/2$, where $\varepsilon > 0$. Thus, for example, to estimate the second derivative of solution u we need to assume

$$\|g\|_{\mathbb{H}_2^{\gamma_0}(T, \ell_2)} < \infty \quad \text{if } \gamma \neq 1/2, \quad \text{and} \quad \|g\|_{\mathbb{H}_2^\varepsilon(T, \ell_2)} < \infty \quad \text{if } \gamma = 1/2.$$

Recall $\gamma_0 = (2\gamma - 1)_+/\beta < 2$, which is defined in (3.6).

We first consider the equation

$$\partial_t^\beta u(t, x) = \Delta u(t, x) + f(t, x) + \sum_{k=1}^\infty \partial_t^\gamma \int_0^t g^k(s, x) dW_s^k. \quad (3.18)$$

Note that by letting $\beta \rightarrow 1$ and $\gamma \rightarrow 1$ we get the classical stochastic partial differential equations.

Lemma 3.9. *Let $f \in \mathbb{L}_2(T)$, $g \in \mathbb{H}_0^\infty(T, \ell_2)$ and $u \in \mathbb{H}_2^2(T)$. Then u satisfies (3.18) with initial data $u_0 \in U_2^1$ in the sense distributions (see Definition 3.4) if and only if*

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}^d} p(t, x - y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} q(t - s, x - y) f(s, y) dy ds \\ &+ \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} P_{\beta, \gamma}(t - s, x - y) g^k(s, y) dy dW_s^k. \end{aligned} \quad (3.19)$$

Proof. Suppose u satisfies (3.18). Recall that for the solution of the (deterministic) equation

$$\partial_t^\beta \bar{u} = \Delta \bar{u} + f, \quad u(0) = u_0$$

is given by the formula

$$\bar{u}(t, x) := \int_{\mathbb{R}^d} p(t, x - y) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} q(t - s, x - y) f(s, y) dy ds. \quad (3.20)$$

In fact, in [2, Section 5.2] the representation (3.20) is proved for sufficiently smooth f . This and the estimate of the solution obtained in [14, Theorem 3.1] allow us to use an approximation (see the proof of Theorem 3.10) and get (3.20) for general $f \in \mathbb{L}_2(T)$.

Thus by considering $u - \bar{u}$, where \bar{u} is defined above, we may assume without loss of generality that $u_0 = 0$ and $f = 0$. Suppose first $\beta \leq \gamma$. Set $a = \gamma - \beta$,

$$v(t, x) := \sum_{k=1}^{\infty} \int_0^t g^k(s, y) dW_s^k,$$

and

$$w(t, x) := D_t^a v(t, x) = \frac{1}{\Gamma(1-a)} \sum_{k=1}^{\infty} \int_0^t (t-s)^{-a} g^k(s, x) dW_s^k.$$

Then $u - w$ satisfies the following fractional diffusion equation

$$\partial_t^\beta (u - w) = \Delta u = \Delta(u - w) + \Delta w, \quad (u - w)(0) = 0.$$

Thus by (3.20) with Δw in place of f , we have

$$u(t, x) = w(t, x) + \int_0^t \int_{\mathbb{R}^d} q(t - s, x - y) \Delta w(s, y) dy ds.$$

Nota that for any $s < t$

$$\begin{aligned} \int_{\mathbb{R}^d} q(t - s, x - y) \Delta w(s, y) dy &= \int_{\mathbb{R}^d} \Delta q(t - s, x - y) D_t^a v(s, y) dy \\ &= \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p(t - s, x - y) D_t^a v(s, y) dy \\ &= \frac{\partial}{\partial t} \int_{\mathbb{R}^d} p(t - s, x - y) D_t^a v(s, y) dy. \end{aligned}$$

Indeed, the first equality is from Lemma 2.2(iv) and the integration by parts, the second equality is from Lemma 2.4 and the third equality is from Lemma 2.2(v). Therefore $u(t, x)$ is equal to

$$\begin{aligned} &D_t^a v(t, x) + \int_0^t \frac{\partial}{\partial t} \int_{\mathbb{R}^d} p(t - s, x - y) D_t^a v(s, y) dy ds \\ &= \frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^d} p(t - s, x - y) D_t^a v(s, y) dy ds \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-a)} \sum_{k=1}^{\infty} \frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y) \int_0^s (s-r)^{-a} g^k(r, y) dW_r^k dy ds \\
&= \frac{1}{\Gamma(1-a)} \sum_{k=1}^{\infty} \frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^d} p(s, x-y) \int_0^{t-s} (t-s-r)^{-a} g^k(r, y) dW_r^k dy ds.
\end{aligned}$$

Hence it is enough to prove

$$\begin{aligned}
&\frac{1}{\Gamma(1-a)} \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} p(s, x-y) \int_0^{t-s} (t-s-r)^{-a} g^k(r, y) dW_r^k dy ds \\
&= \sum_{k=1}^{\infty} \int_0^t \int_0^s \int_{\mathbb{R}^d} D_t^a p(s-r, x-y) g^k(r, y) dy dW_r^k ds. \tag{3.21}
\end{aligned}$$

The latter equals

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} \int_0^{t-r} D_t^a p(s, x-y) ds g^k(r, y) dy dW_r^k \\
&= \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} I^{1-a} p(t-r, x-y) g^k(r, y) dy dW_r^k \tag{3.22} \\
&= \frac{1}{\Gamma(1-a)} \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} \int_0^{t-r} (t-r-s)^{-a} p(s, x-y) ds g^k(r, y) dy dW_r^k.
\end{aligned}$$

For (3.22) above we used the fact that $\int_0^t D_t^a p ds = I^{1-a} p$. We thus get (3.21) using the stochastic Fubini's theorem (see [6]).

We now consider the case $\beta \geq \gamma$. Put $a = \beta - \gamma$ and define

$$v(t, x) = \int_0^t g^k(s, x) dW_s^k, \quad w(t, x) = I^a v(t, x) = \frac{1}{a\Gamma(a)} \int_0^t (t-s)^a g^k(s, x) dW_s^k.$$

From this point on it is enough to repeat the case $\beta \leq \gamma$. Indeed, following the previous steps, we get

$$u(t, x) = \frac{1}{a\Gamma(a)} \sum_{k=1}^{\infty} \frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^d} p(s, x-y) \int_0^{t-s} (t-s-r)^a g^k(r, y) dW_r^k dy ds.$$

Note that

$$\begin{aligned}
&\sum_{k=1}^{\infty} \int_0^t \int_0^s \int_{\mathbb{R}^d} I^a p(s-r, x-y) g^k(r, y) dy dW_r^k ds \\
&= \sum_{k=1}^{\infty} \int_0^t \int_0^{t-r} \int_{\mathbb{R}^d} I^a p(s, x-y) g^k(r, y) dy ds dW_r^k \\
&= \frac{1}{a\Gamma(a)} \sum_{k=1}^{\infty} \int_0^t \int_0^{t-r} \int_{\mathbb{R}^d} (t-s-r)^a p(s, x-y) g^k(r, y) dy ds dW_r^k.
\end{aligned}$$

This clearly proves the case $\gamma \geq \beta$.

On the other hand, going backward of the above equalities one easily finds that if u is given as in (3.19), then it satisfies (3.18). The proof of the lemma is now complete. \square

Recall $\gamma_0 = (2\gamma - 1)_+/\beta < 2$. Fix $\varepsilon_0 \in (0, 1)$ and define

$$\sigma_0 := \gamma_0 + (\varepsilon_0 1_{\gamma=1/2}) = \begin{cases} (2\gamma - 1)/\beta & \text{if } \gamma > 1/2 \\ \varepsilon_0 & \text{if } \gamma = 1/2 \\ 0 & \text{if } \gamma < 1/2. \end{cases} \quad (3.23)$$

Theorem 3.10. *For any $f \in \mathbb{H}_2^\sigma(T)$, $g \in \mathbb{H}_2^{\sigma+\sigma_0}(T, \ell_2)$ and $u_0 \in U_2^{\sigma+1}$, equation (3.18) has a unique solution $u \in \mathcal{H}_2^{\sigma+2}(T)$, and for this solution we have*

$$\|u\|_{\mathbb{H}_2^{\sigma+2}(T)} \leq N \left(\|u_0\|_{U_2^{\sigma+1}} + \|f\|_{\mathbb{H}_2^\sigma(T)} + \|g\|_{\mathbb{H}_2^{\sigma+\sigma_0}(T, \ell_2)} \right), \quad (3.24)$$

where N depends only on d and T .

Proof. Without loss of generality we only need to prove the case $\sigma = 0$.

For the deterministic equation, the theorem including the estimate is proved in [14]. Thus the uniqueness for equation (3.18) easily follows.

Define w as in (3.20). Then by considering $u - w$, we may assume without loss of generality that $u_0 = 0$ and $f = 0$.

First, assume $g \in \mathbb{H}_0^\infty(T, \ell_2)$. Then by Lemmas 3.8 and 3.9, equation (3.18) has a unique solution $u \in \mathbb{H}_2^2(T)$ and estimate (3.24) holds.

For general $g \in \mathbb{H}_2^{\sigma_0}(T, \ell_2)$, take a sequence of $g_n \in \mathbb{H}_0^\infty(T, \ell_2)$ so that $g_n \rightarrow g$ in $\mathbb{H}_2^{\sigma_0}(T, \ell_2)$. Define u_n as the solution of equation (3.18) with g_n in place of g , that is,

$$(I^{1-\beta}u_n)(t) = \int_0^t \Delta u_n(s) ds + \sum_{k=1}^{\infty} (I^{1-\gamma} \int_0^\cdot g_n^k(s) dW_s^k)(t). \quad (3.25)$$

Then by Lemmas 3.8 and 3.9

$$\begin{aligned} \|u_n\|_{\mathbb{H}_2^2(T)} &\leq N \|g_n\|_{\mathbb{H}_2^{\sigma_0}(T, \ell_2)}, \\ \|u_n - u_m\|_{\mathbb{H}_2^2(T)} &\leq N \|g_n - g_m\|_{\mathbb{H}_2^{\sigma_0}(T, \ell_2)}. \end{aligned} \quad (3.26)$$

Thus $u_n \rightarrow u$ in $\mathbb{H}_2^2(T)$ for some u . Letting $n \rightarrow \infty$ in (3.25) and using Remark 3.2, we see that u is a solution of (3.18). Also we easily get (3.24) from (3.26). The theorem is proved. \square

The following lemma is taken from [13, Corollary 2].

Lemma 3.11. *(Gronwall's lemma) Suppose $b > 0$ and $a(t)$ is a nonnegative nondecreasing locally integrable function on $[0, T)$, and suppose $\eta(t)$ is nonnegative locally integrable on $[0, T)$ with*

$$\eta(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} \eta(s) ds, \quad \forall t < T.$$

Then it holds that

$$\eta(t) \leq a(t) E_\beta(b\Gamma(\beta)t^\beta).$$

4. SPDE of divergence form type

In this section, we study the equation of divergence type

$$\begin{aligned} \partial_t^\beta u &= D_i [a^{ij} u_{x^j} + b^i u + f^i(u)] + cu + h(u) \\ &\quad + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t (\sigma^{ijk} u_{x^i x^j} + \mu^{ik} u_{x^i} + \nu^k u + g^k(u)) dW_s^k \end{aligned} \quad (4.1)$$

where the coefficients $a^{ij}, b^i, c, \sigma^{ijk}, \mu^{ik}, \nu^k$ are functions depending on (ω, t, x) and the functions f^i, h, g^k depend on (ω, t, x) and the unknown u .

For a ℓ_2 -valued continuous function v in \mathbb{R}^d , we define the space $C^\alpha(\ell_2)$, $\alpha \in [0, 1]$ by the norm

$$|v|_{C^\alpha(\ell_2)} = \sup_x |v|_{\ell_2} + \sup_{x \neq y} \frac{|v(x) - v(y)|_{\ell_2}}{|x - y|^\alpha}.$$

Assumption 4.1. (i) The coefficients $a^{ij}, b^i, c, \sigma^{ijk}, \mu^{ik}, \nu^{ik}$ are $\mathcal{P} \otimes \mathcal{B}([0, T] \times \mathbb{R}^d)$ -measurable.
(ii) There exist constants $\delta, K_1 > 0$ so that for any $\xi \in \mathbb{R}^d$

$$\delta |\xi|^2 \leq a^{ij} \xi_i \xi_j \leq K_1 |\xi|^2 \quad \forall i, j, \omega, t, x.$$

$$|b^i| + |c| + |\sigma^{ij}|_{\ell_2} + |\mu^i|_{\ell_2} + |\nu|_{\ell_2} \leq K_1 \quad \forall i, j, \omega, t, x. \quad (4.2)$$

(iii) $\sigma^{ijk} = 0$ if $\gamma \geq 1/2$, and $\mu^{ik} = 0$ if $\gamma \geq 1/2 + \beta/2$ for every i, j, k, ω, t, x .

Recall that σ_0 is the constant defined in (3.23).

Assumption 4.2. (i) There exist constant $\kappa, K_2 > 0$,

$$|\sigma^{ij}(t, \cdot)|_{C^1(\ell_2)} + |\mu^i(t, \cdot)|_{C^{|\sigma_0-1|+\kappa}(\ell_2)} + |\nu(t, \cdot)|_{C^{|\sigma_0-1|+\kappa}(\ell_2)} \leq K_2, \quad \forall i, j, \omega, t.$$

(ii) For any $\varepsilon > 0$ there exists $K_3 = K_3(\varepsilon, T)$ so that

$$\begin{aligned} & \|f^i(t, \cdot, u(\cdot)) - f^i(t, \cdot, v(\cdot))\|_{L_2} + \|h(t, \cdot, u(\cdot)) - h(t, \cdot, v(\cdot))\|_{H_2^{-1}} \\ & + \|g(t, \cdot, u(\cdot)) - g(t, \cdot, v(\cdot))\|_{H_2^{-1+\sigma_0}(\ell_2)} \leq \varepsilon \|u\|_{H_2^1} + K_3 \|u\|_{L_2}, \end{aligned}$$

for any $u, v \in H_2^1$ and ω, t .

See Example 5.3 for examples satisfying Assumption 4.2(ii).

Denote

$$f_0^i = f^i(t, x, 0), \quad h_0 = h(t, x, 0), \quad g_0 = g(t, x, 0).$$

We will use a well-known inequality (eg. [5, Lemma 5.2])

$$\|au\|_{H_2^\gamma} \leq N(\sigma, d) |a|_{C^\gamma} \|u\|_{H_2^\sigma}, \quad (4.3)$$

where $\gamma \geq |\sigma|$ is σ is an integral, and otherwise $\gamma > |\sigma|$.

The following is the one of the two main results of this paper.

Theorem 4.3. *Suppose Assumptions 4.1 and 4.2 hold. There exists $\kappa_0 > 0$ depending only on K, γ, β, d, T so that if $\sup_{\omega, i, j, t \leq T} |\sigma^{ij}(t, \cdot)|_{C^1(\ell_2)} \leq \kappa_0$ then equation (4.1) with initial data $u_0 \in U_2^0$ has a unique solution $u \in \mathbb{H}_2^1(T)$, and for this solution*

$$\|u\|_{\mathcal{H}_2^1(T)}^2 \leq N \left(\mathbb{E} \|u_0\|_{L_2}^2 + \|f_0^i\|_{L_2(T)}^2 + \|h_0\|_{\mathbb{H}_2^{-1}(T)}^2 + \|g_0\|_{\mathbb{H}_2^{-1+\sigma_0}(\ell_2)}^2 \right), \quad (4.4)$$

where N depends only on $\gamma, \beta, \delta, d, K$ and T .

Proof. A: Linear case. Let f^i, h and g^k depend only on (ω, t, x) . Due to the method of continuity and solvability result of Lemma 3.10, it is enough to show that there exists $\kappa_0 > 0$ so that if $|\sigma^{ij}(t, \cdot)|_{C^1(\ell_2)} \leq \kappa_0$ and $u \in \mathcal{H}_2^1(T)$ is a solution of (4.1), then the estimate (4.4) holds. We refer the reader to the proof of [5, Theorem 5.1] for details.

Step 1. Assume $b^i = c = \mu^{ik} = \nu^k = 0$.

By Theorem 3.10, the equation

$$\partial_t^\beta v = \Delta v + D_i f^i + h + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t (\sigma^{ijk} u_{x^i x^j} + g^k) dW_s^k$$

with initial data u_0 has a unique solution $v \in \mathcal{H}_2^1(T)$, and

$$\|v\|_{\mathbb{H}_2^1(T)}^2 \leq N \left(\mathbb{E}\|u_0\|_{L_2}^2 + \|D_i f^i + h\|_{\mathbb{H}_2^{-1}(T)}^2 + \|\sigma^{ij} u_{x^i x^j} + g\|_{\mathbb{H}_2^{-1+\sigma_0}(T, \ell_2)}^2 \right). \quad (4.5)$$

Note that for each ω , $\bar{u} = u - v$ satisfies the deterministic equation

$$\partial_t^\beta \bar{u} = D_i(a^{ij} u_{x^j}) - \Delta v = D_i(a^{ij} \bar{u}_{x^j} + a^{ij} v_{x^j} - v_{x^i}) = D_i(a^{ij} \bar{u}_{x^j} + \bar{f}^i), \quad \bar{u}(0) = 0$$

where $\bar{f}^i = \sum_{j=1}^d a^{ij} v_{x^j} - v_{x^i}$. By a result for the deterministic equations (see [14]),

$$\|\bar{u}\|_{\mathbb{H}_2^1(T)} \leq N \|\bar{f}^i\|_{L_2(T)} \leq N \|v\|_{\mathbb{H}_2^1(T)}.$$

Note that $\sigma^{ij} = 0$ if $\gamma \geq 1/2$. If $\gamma < 1/2$ then $-1 + \sigma_0 = -1$ and so by (4.3) for every $t \leq T$

$$\begin{aligned} \|\sigma^{ij}(t, \cdot) u_{x^i x^j}(t, \cdot)\|_{H_2^{-1}(\ell_2)} &\leq N |\sigma^{ij}(t, \cdot)|_{C^1(\ell_2)} \|u_{x^i x^j}(t, \cdot)\|_{H_2^{-1}} \\ &\leq N \sup_{\omega, i, j, t \leq T} |\sigma^{ij}(t, \cdot)|_{C^1(\ell_2)} \|u(t, \cdot)\|_{H_2^1}. \end{aligned}$$

This and (4.5) certainly lead to

$$\begin{aligned} \|u\|_{\mathbb{H}_2^1(T)}^2 &\leq N_0 \sup_{\omega, i, j, t \leq T} |\sigma^{ij}(t, \cdot)|_{C^1(\ell_2)} \|u\|_{\mathbb{H}_2^1(T)}^2 \\ &\quad + N_0 \left(\|u_0\|_{U_2^0}^2 + \|D_i f^i + h\|_{\mathbb{H}_2^{-1}(T)}^2 + \|g\|_{\mathbb{H}_2^{-1+\sigma_0}(T, \ell_2)}^2 \right), \end{aligned} \quad (4.6)$$

where N_0 depends only on $\beta, \gamma, \delta, d, T$ and K . Note that $\|D_i f^i\|_{H_2^{-1}} \leq N \|f^i\|_{L_2}$. Hence for the desired estimate it is enough to take

$$\kappa_0 = (2N_0)^{-1/2}.$$

Step 2. Take κ_0 from *Step 1*, and assume $\sup_{\omega, i, j, t \leq T} |\sigma^{ij}(t, \cdot)|_{C^1(\ell_2)} \leq \kappa_0$. Then by the result of *Step 1*, for each $t \leq T$,

$$\|u\|_{\mathbb{H}_2^1(t)}^2 \leq N \|b^i u + f^i\|_{L_2(t)}^2 + N \|cu + h\|_{\mathbb{H}_2^{-1}(t)}^2 + N \|\mu^i u_{x^i} + \nu u + g\|_{\mathbb{H}_2^{-1+\sigma_0}(t, \ell_2)}^2. \quad (4.7)$$

Note that, by (4.2)

$$\|b^i u\|_{L_2} + \|cu\|_{H_2^{-1}} \leq \|b^i u\|_{L_2} + \|cu\|_{L_2} \leq N \|u\|_{L_2} \leq \varepsilon \|u\|_{H_2^1} + N \|u\|_{H_2^{-1}}.$$

Also, since $-1 + \sigma_0 < 1$, by a Sobolev embedding theorem and (4.3), for any $\varepsilon > 0$

$$\|\nu u\|_{H_2^{\sigma_0-1}} \leq N |\nu|_{C^{|\sigma_0-1|+\kappa}(\ell_2)} \|u\|_{H_2^{\sigma_0-1}} \leq N \varepsilon \|u\|_{H_2^1} + N(\varepsilon) \|u\|_{H_2^{-1}}.$$

By the assumption, $\nu^i = 0$ unless $-1 + \sigma_0 < 0$. Therefore,

$$\|\mu^i u_{x^i}\|_{H_2^{-1+\sigma_0}} \leq N |\mu^i|_{C^{|\sigma_0-1|+\kappa}(\ell_2)} \|u_{x^i}\|_{H_2^{-1+\sigma_0}} \leq N \|u\|_{H_2^{\sigma_0}} \leq N \varepsilon \|u\|_{H_2^1} + N(\varepsilon) \|u\|_{H_2^{-1}}.$$

Taking sufficiently small $\varepsilon > 0$ and using (4.7), we get for any $t \leq T$,

$$\|u\|_{\mathbb{H}_2^1(t)}^2 \leq N \|u\|_{\mathbb{H}_2^{-1}(t)}^2 + N (\|u_0\|_{U_2^0}^2 + \|f^i\|_{L_2(t)}^2 + \|h\|_{\mathbb{H}_2^{-1}(t)}^2 + \|g\|_{\mathbb{H}_2^{-1+\sigma_0}(t, \ell_2)}^2). \quad (4.8)$$

Since

$$\mathbb{D}u = D_i(a^{ij} u_{x^j} + b^i u + f) + cu + h, \quad \mathbb{S}u = \sigma^{ij} u_{x^i x^j} + \mu^i u_{x^i} + \nu u + g$$

and $D_i : H_2^\gamma \rightarrow H_2^{\gamma-1}$ is a bounded operator, we have

$$\|\mathbb{D}u\|_{\mathbb{H}_2^{-1}(t)}^2 \leq N \left(\|u\|_{\mathbb{H}_2^1(t)}^2 + \|f\|_{L_2(t)}^2 \right) \quad \text{and} \quad \|\mathbb{S}u\|_{\mathbb{H}_2^{-1}(t, \ell_2)}^2 \leq N \left(\|u\|_{\mathbb{H}_2^1(t)}^2 + \|g\|_{\mathbb{H}_2^{-1}(t, \ell_2)}^2 \right). \quad (4.9)$$

This, (4.8) and Proposition 3.7 (with $\sigma = -1$) yield

$$(k_{1-\beta} * \mathbb{E}\|u\|_{H_2^{-1}}^2)(t) \leq N (\|u_0\|_{U_2^0}^2 + \|f^i\|_{L_2(t)}^2 + \|h\|_{\mathbb{H}_2^{-1}(t)}^2 + \|g\|_{\mathbb{H}_2^{-1+\sigma_0}(t, \ell_2)}^2 + \|u\|_{\mathbb{H}_2^{-1}(t)}^2).$$

Denote $\eta(t) = \mathbb{E} \int_0^t \|u(s, \cdot)\|_{H_2^{-1}}^2 ds$. Then taking the convolution with k_β (recall $k_\beta * k_{1-\beta} = 1$), we get

$$\eta(t) \leq NM + N(k_\beta * \eta)(t), \quad (4.10)$$

where

$$M := \|u_0\|_{U_2^0}^2 + \|f^i\|_{L_2(T)}^2 + \|h\|_{\mathbb{H}_2^{-1}(T)}^2 + \|g\|_{\mathbb{H}_2^{-1+\sigma_0}(T, \ell_2)}^2.$$

Consequently (4.8), (4.10) and Gronwall's lemma (Lemma 3.11) finish the proof for the linear case.

B: Non-linear case. The proof is identical to that of the non-divergence type case. See the proof of Theorem 5.4 below (it is enough to replace σ by -1). \square

5. SPDE of non-divergence form type

5.1. L_2 -theory for fractional time SPDE of non-divergence form type. In this subsection, we study the equation of non-divergence type

$$\begin{aligned} \partial_t^\beta u &= (a^{ij} u_{x^i x^j} + b^i u + cu + f(u)) \\ &\quad + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t (\sigma^{ijk} u_{x^i x^j} + \mu^{ik} u_{x^i} + \nu^k u + g^k(u)) dW_s^k \end{aligned} \quad (5.1)$$

where the coefficients $a^{ij}, b^i, c, \sigma^{ijk}, \mu^{ik}, \nu^k$ are functions depending on (ω, t, x) and the functions f, g^k depend on (ω, t, x) and the unknown u . Recall that σ_0 is the constant defined by (3.23)

Assumption 5.1. (i) The coefficients a^{ij} are uniformly continuous in x , that is for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$|a^{ij}(t, x) - a^{ij}(t, y)| < \varepsilon, \quad \forall i, j, \omega, t$$

whenever $|x - y| < \delta$.

(ii) Hölder continuity of a^{ij} when $\sigma \neq 0$: if $\sigma \neq 0$, there exists constants $\kappa, K_1 > 0$ so that

$$|a^{ij}(t, \cdot)|_{C^{|\sigma|+\kappa}} < K_1, \quad \forall i, j, \omega, t. \quad (5.2)$$

(iii) For any i, j, ω and t

$$|b^i(t, \cdot)|_{C^{|\sigma|+\kappa}} + |c(t, \cdot)|_{C^{|\sigma|+\kappa}} + |\sigma^{ij}(t, \cdot), \mu^i(t, \cdot), \nu(t, \cdot)|_{C^{|\sigma+\sigma_0|+\kappa}(\ell_2)} \leq K_2 < \infty. \quad (5.3)$$

(iv) $|\sigma^{ij}(t, x)|_{\ell_2} \leq \kappa_0$, where κ_0 is the constant in Theorem 4.3.

(v) For any $\varepsilon > 0$ there exists $K_3 = K_3(\varepsilon)$ so that

$$\|f(t, \cdot, u(\cdot)) - f(t, \cdot, v(\cdot))\|_{H_2^\sigma} + \|g(t, \cdot, u(\cdot)) - g(t, \cdot, v(\cdot))\|_{H_2^{\sigma+\sigma_0}} \leq \varepsilon \|u\|_{H_2^{\sigma+2}} + K_3 \|u\|_{H_2^{\sigma+1}}, \quad (5.4)$$

for any $u, v \in H_2^{\sigma+2}$.

Remark 5.2. If σ is integer then one can slightly weaken (5.2) and (5.3) and take $\kappa = 0$ as is done in [5].

Example 5.3. (i) Let $\delta := \sigma + 2 - d/2 > 0$ and $f_0 = f_0(x) \in H_2^\sigma$. Take

$$f(x, u) = f_0(x) \sup_x |u|.$$

Then by a Sobolev embedding

$$\begin{aligned} \|f(u) - f(v)\|_{H_2^\sigma} &\leq \|f_0\|_{H_2^\sigma} \sup_x |u - v| \leq N \|u - v\|_{H_2^{\sigma+2-\delta/2}} \\ &\leq \varepsilon \|u - v\|_{H_2^{\sigma+2}} + K \|u - v\|_{H_2^\sigma}. \end{aligned} \quad (5.5)$$

(ii) Fix $\varepsilon > 0$ and take $\delta \in (0, 1)$ and a random $C^{|\sigma|+\varepsilon}$ -function $a(t, x)$. Let

$$f(t, x, u) = a(t, x)(-\Delta)^\delta u, \quad |a(t, \cdot)|_{C^{|\sigma|+\varepsilon}} \leq K, \quad \forall \omega, t.$$

Then the argument used to prove (5.5) easily leads to (5.4).

Denote

$$f_0 = f(t, x, 0), \quad g_0 = g(t, x, 0).$$

Here is the second main result of this paper.

Theorem 5.4. *Let $\sigma \in \mathbb{R}$ and Assumptions 4.1 and 5.1 hold. Then for any $f \in \mathbb{H}_2^\sigma(T)$, $g \in \mathbb{H}_2^{\sigma+\sigma_0}(T, \ell_2)$ and $u_0 \in U_2^{\sigma+1}$, equation (5.1) admits a unique solution $u \in \mathcal{H}_2^{\sigma+2}(T)$, and for this solution we have*

$$\|u\|_{\mathbb{H}_2^{\sigma+2}(T)} \leq N \left(\|u_0\|_{U_2^{\sigma+1}} + \|f_0\|_{\mathbb{H}_2^\sigma(T)} + \|g_0\|_{\mathbb{H}_2^{\sigma+\sigma_0}(T, \ell_2)} \right), \quad (5.6)$$

where N depends only on $d, \beta, \gamma, \delta, K$ and T .

Proof. By considering $u - v$ if needed, where v is the solution of $\partial_t^\beta v = \Delta v$ with $v(0) = u_0$, we may assume without loss of generality that $u_0 = 0$.

A: Linear case. Let f^i , h and g^k depend only on (ω, t, x) . Due to the method of continuity we only need to prove that the estimate (5.6) holds given that a solution already exists.

Step 1. Assume that all the coefficients are independent of x , so that equation (5.1) is of type (4.1). By applying the operator $(1 - \Delta)^{(\sigma+1)/2}$ to equation (5.1), one can simplify the problem to the case $\sigma = -1$. In this case all the claims follow from Theorem 4.4.

Step 2. Next, we weaken the condition in *Step 1* by proving that there exists a $\varepsilon_1 \in (0, \kappa_0]$ so that the theorem holds if

$$|a^{ij}(t, x) - a^{ij}(t, y)| + |\sigma^{ij}(t, x) - \sigma^{ij}(t, y)|_{\ell_2} \leq \varepsilon_1 \quad \forall i, j, \omega, t, x, y. \quad (5.7)$$

Fix $x_0 \in \mathbb{R}^d$ and denote

$$a_0^{ij}(t, x) = a^{ij}(t, x_0), \quad \sigma_0^{ij}(t, x) = \sigma^{ij}(t, x_0).$$

Note that equation (5.1) can be written as

$$\partial_t^\beta u = \left(a_0^{ij} u_{x^i x^j} + \bar{f} \right) + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t (\sigma_0^{ijk} u_{x^i x^j} + \bar{g}^k) dW_s^k,$$

where

$$\begin{aligned} \bar{f} &:= (a^{ij} - a_0^{ij}) u_{x^i x^j} + b^i u_{x^i} + cu + f, \\ \bar{g}^k &:= (\sigma^{ijk} - \sigma_0^{ijk}) u_{x^i x^j} + \mu^{ik} u_{x^i} + \nu^k u + g^k. \end{aligned}$$

Note that the coefficients a_0^{ij} and σ_0^{ij} are independent of x . By the result of *Step 1*, for each $t \leq T$,

$$\|u\|_{\mathbb{H}_2^{\sigma+2}(t)} \leq N \left(\|u_0\|_{U_2^{\sigma+1}} + \|\bar{f}\|_{\mathbb{H}_2^\sigma(t)} + \|\bar{g}\|_{\mathbb{H}_2^{\sigma+\sigma_0}(t, \ell_2)} \right). \quad (5.8)$$

To estimate \bar{f} and \bar{g} we use the following well known embedding result: for $0 \leq \alpha_1 \leq \alpha_2$ and $\alpha_2 > 0$

$$|v|_{C^{\alpha_1}} \leq N |v|_{C^0}^{1-\alpha_3} |v|_{C^{\alpha_2}}^{\alpha_3}, \quad \alpha_3 := \frac{\alpha_1}{\alpha_2}. \quad (5.9)$$

If $\sigma = 0$, then

$$\|(a^{ij}(t, \cdot) - a_0^{ij}(t)) u_{x^i x^j}(t, \cdot)\|_{L_2} \leq N \sup_x |a^{ij}(t, \cdot) - a_0^{ij}(t)| \cdot \|u(t, \cdot)\|_{H_2^2},$$

and otherwise, by first using (4.3) and then taking $\alpha_1 = |\sigma| + \kappa/2$ and $\alpha_2 = |\sigma| + \kappa$ in (5.9),

$$\begin{aligned} \|(a^{ij}(t, \cdot) - a_0^{ij}(t))u_{x^i x^j}(t, \cdot)\|_{H_2^\sigma} &\leq N|a^{ij}(t, \cdot) - a_0^{ij}(t)|_{C^{|\sigma|+\kappa/2}} \cdot \|u(t, \cdot)\|_{H_2^{\sigma+2}} \\ &\leq N \sup_x |a^{ij}(t, \cdot) - a_0^{ij}(t)|^\delta \cdot \|u(t, \cdot)\|_{H_2^{\sigma+2}}. \end{aligned}$$

where

$$\delta := 1 - \frac{|\sigma| + \kappa/2}{|\sigma| + \kappa} > 0.$$

The term $\|(\sigma^{ij}(t, \cdot) - \sigma_0^{ij}(t))u_{x^i x^j}(t, \cdot)\|_{H_2^{\sigma+\sigma_0}}$ and others can be handled similarly. For instance, by (4.3)

$$\|b^i(t, \cdot)u_{x^i}(t, \cdot)\|_{H_2^\sigma} \leq N|b^i(t, \cdot)|_{C^{|\sigma|+\kappa}} \cdot \|u(t, \cdot)\|_{H_2^{\sigma+1}} \leq \varepsilon\|u(t, \cdot)\|_{H_2^{\sigma+2}} + N\|u(t, \cdot)\|_{H_2^\sigma},$$

and, since $\mu^i = 0$ unless $\sigma_0 < 1$,

$$\begin{aligned} \|\mu^i(t, \cdot)u_{x^i}(t, \cdot)\|_{H_2^{\sigma+\sigma_0}(\ell_2)} &\leq N|\mu(t, \cdot)|_{C^{|\sigma+\sigma_0|+\kappa}(\ell_2)}\|u(t, \cdot)\|_{H_2^{\sigma+\sigma_0+1}} \\ &\leq \varepsilon\|u(t, \cdot)\|_{H_2^{\sigma+2}} + N\|u(t, \cdot)\|_{H_2^\sigma}. \end{aligned}$$

Hence, from (5.8) it follows

$$\begin{aligned} \|u\|_{\mathbb{H}_2^{\sigma+2}(t)} &\leq N \left(\sup_{x, t \leq T} |a^{ij} - a_0^{ij}|^\delta + \sup_{x, t \leq T} |\sigma^{ij} - \sigma_0^{ij}|_{\ell_2}^\delta + \varepsilon \right) \cdot \|u\|_{\mathbb{H}_2^{\sigma+2}(t)} \\ &\quad + N\|u\|_{\mathbb{H}_2^\sigma(t)} + N \left(\|u_0\|_{U_2^{\sigma+1}} + \|f\|_{\mathbb{H}_2^\sigma(t)} + \|g\|_{\mathbb{H}_2^{\sigma+\sigma_0}(t, \ell_2)} \right). \end{aligned}$$

Take $\varepsilon, \varepsilon_1 > 0$ so that $\varepsilon, \varepsilon_1 < (4N)^{-1}$. If we assume (5.7) then for each $t \leq T$,

$$\|u\|_{\mathbb{H}_2^{\sigma+2}(t)} \leq N\|u\|_{\mathbb{H}_2^\sigma(t)} + \left(\|u_0\|_{U_2^{\sigma+1}} + \|f\|_{\mathbb{H}_2^\sigma(t)} + \|g\|_{\mathbb{H}_2^{\sigma+\sigma_0}(t, \ell_2)} \right). \quad (5.10)$$

Just as (4.9) and the rest of the argument in the proof of Theorem 4.3, this and Gronwall's lemma (Lemma 3.11) lead to the desired estimate.

Step 3. General linear case without condition (5.7). Extension of *Step 2* to the general case is quite straightforward and can be found for example in the proof of [5, Theorem 5.1]. One introduces a partition of unity $\{\zeta_n : n = 1, 2, \dots\}$ of $C_0^\infty(\mathbb{R}^d)$ -functions so that (5.7) holds on each support of ζ_n . Then one estimates $u\zeta_n$ using the result of *Step 2* and by summing up these estimate one easily gets (5.10), which is sufficient for our estimate.

B: Non-linear case. We modify the proof of [5, Theorem 5.1]. Recall that $\mathcal{H}_{2,0}^{\sigma+2}(T)$ is defined in (3.11). For each $u \in \mathcal{H}_{2,0}^{\sigma+2}(T)$ consider the equation

$$\begin{aligned} \partial_t^\beta v &= (a^{ij}v_{x^i x^j} + b^i v + cv + f(u)) \\ &\quad + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t (\sigma^{ijk}v_{x^i x^j} + \mu^{ik}v_{x^i} + \nu^k v + g^k(u)) dW_s^k \end{aligned}$$

with initial data $v(0) = 0$. By the above results, this equation has a unique solution $v \in \mathcal{H}_{2,0}^{\sigma+2}(T)$. By denoting $v = \mathcal{R}u$ we can define an operator $\mathcal{R} : \mathcal{H}_{2,0}^{\sigma+2}(T) \rightarrow \mathcal{H}_{2,0}^{\sigma+2}(T)$.

Note that due to the interpolation $\|\xi\|_{H_2^{\sigma+1}} \leq \varepsilon\|\xi\|_{H_2^{\sigma+2}} + N\|\xi\|_{H_2^\sigma}$, (5.4) is equivalent to

$$\|f(t, \cdot, u(\cdot)) - f(t, \cdot, v(\cdot))\|_{H_2^\sigma} + \|g(t, \cdot, u(\cdot)) - g(t, \cdot, v(\cdot))\|_{H_2^{\sigma+\sigma_0}} \leq \varepsilon\|u\|_{H_2^{\sigma+2}} + K\|u\|_{H_2^\sigma} \quad (5.11)$$

for some $K = K(\varepsilon) > 0$.

By the results for the linear case and (5.11) and Proposition 3.7, for each $t \leq T$,

$$\begin{aligned} \|\mathcal{R}u - \mathcal{R}v\|_{\mathcal{H}_2^{\sigma+2}(t)}^2 &\leq N\|f(u) - f(v)\|_{\mathbb{H}_2^\sigma(t)}^2 + N\|g(u) - g(v)\|_{\mathbb{H}_2^{\sigma+\sigma_0}(t, \ell_2)}^2 \\ &\leq N\varepsilon^2\|u - v\|_{\mathcal{H}_2^{\sigma+2}(t)}^2 + NK^2\|u - v\|_{\mathbb{H}_2^\sigma(t)}^2 \\ &\leq N_0\varepsilon^2\|u - v\|_{\mathcal{H}_2^{\sigma+2}(t)}^2 + N_1 \int_0^t (t-s)^{-1+\beta}\|u - v\|_{\mathcal{H}_2^{\sigma+2}(s)}^2 ds, \end{aligned}$$

where N_1 depends also on ε . Next, we fix ε so that $\theta := N_0\varepsilon^2 < 1/4$. Then repeating the above inequality and using the identity

$$\int_0^t (t-s_1)^{-1+\beta} \int_0^{s_1} (s_1-s_2)^{-1+\beta} \cdots \int_0^{s_{n-1}} (s_{n-1}-s_n)^{-1+\beta} ds_n \cdots ds_1 = \frac{\Gamma(\beta)}{\beta\Gamma(n\beta+1)} t^{n\beta},$$

we get

$$\begin{aligned} &\|\mathcal{R}^m u - \mathcal{R}^m v\|_{\mathcal{H}_2^{\sigma+2}(t)}^2 \\ &\leq \sum_{k=0}^m \binom{m}{k} \theta^{m-k} (T^\beta N_1)^k \frac{\Gamma(\beta)}{\beta\Gamma(k\beta+1)} \|u - v\|_{\mathcal{H}_2^{\sigma+2}(t)}^2 \\ &\leq 2^m \theta^m \left[\max_k \left((\theta^{-1} T^\beta N_1)^k \frac{\Gamma(\beta)}{\beta\Gamma(k\beta+1)} \right) \right] \|u - v\|_{\mathcal{H}_2^{\sigma+2}(t)}^2 \\ &\leq \frac{1}{2^m} N_2 \|u - v\|_{\mathcal{H}_2^{\sigma+2}(t)}^2. \end{aligned}$$

For the second inequality above we use $\sum_{k=0}^m \binom{m}{k} = 2^m$. It follows that if m is sufficiently large then \mathcal{R}^m is a contraction in $\mathcal{H}_{2,0}^{\sigma+2}(T)$, and this yields all the claims. The theorem is proved. \square

5.2. An application to SPDE driven by space-time white noise. In this subsection, we consider a SPDE driven by space-time white noise. We consider

$$\partial_t^\beta u = (a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u)) + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t h(u) dB_t \quad (5.12)$$

where the coefficients $a^{ij}, b^i c$ and are functions depending on (ω, t, x) , the functions f and h depends on (ω, t, x) and the unknown u , and B_t is a space-time white noise.

Let $\{\eta^k : k = 1, 2, \dots\}$ be an orthogonal basis of $L_2(\mathbb{R}^d)$. Then (at least formally)

$$B_t = \sum_{k=1}^{\infty} \eta^k W_t^k$$

where $W_t^k := (B_t, \eta^k)_{L_2}$ are independent one dimensional Wiener processes. Hence one can rewrite (5.12) as

$$\partial_t^\beta u = (a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u)) + \sum_{k=1}^{\infty} \partial_t^\gamma \int_0^t h(u) \eta^k dW_t^k.$$

Denote

$$g^k(t, x, u) = h(t, x, u) \eta^k(x).$$

To apply Theorem 5.4, we only need to find σ and conditions on h so that (5.4) holds. The following lemma is a consequence of [5, Lemma 8.4].

Lemma 5.5. *Let $\gamma < -1/2$. Then*

$$\|g(t, \cdot, u(\cdot)) - g(t, \cdot, v(\cdot))\|_{H_2^\gamma(\ell_2)} \leq N \|h(t, \cdot, u(\cdot)) - h(t, \cdot, v(\cdot))\|_{L_2}.$$

The following is an easy consequence of Theorem 5.4 and Lemma 5.5. Recall that $f_0 = f(t, x, 0)$. We also denote $h_0 = h(t, x, 0)$.

Corollary 5.6. *Let*

$$\sigma + \sigma_0 < -1/2, \quad \sigma + 2 > 0. \quad (5.13)$$

Assume

$$|f(t, x, v_1) - f(t, x, v_2)| + |h(t, x, v_1) - h(t, x, v_2)| \leq K|v_1 - v_2| \quad \forall \omega, t, x, v_1, v_2,$$

and Assumptions 4.1 and 5.1 hold with σ satisfying (5.13). Then equation (5.12) with initial data $u_0 \in U_2^{\sigma+1}$ has a unique solution u , and for this solution we have

$$\|u\|_{\mathbb{H}_2^{\sigma+2}(T)} \leq N \left(\|f_0\|_{\mathbb{H}_2^\sigma(T)} + \|h_0\|_{L_2(T)} + \|u_0\|_{U_2^{\sigma+1}} \right).$$

Proof. It is enough to note that, since $\sigma + 2 > 0$,

$$\begin{aligned} & \|f(t, \cdot, u(\cdot)) - f(t, \cdot, v(\cdot))\|_{H_2^\sigma} + \|g(t, \cdot, u(\cdot)) - g(t, \cdot, v(\cdot))\|_{H_2^{\sigma+\sigma_0}(\ell_2)} \\ & \leq N \|u - v\|_{L_2} \leq \varepsilon \|u - v\|_{H_2^{\sigma+2}} + K \|u - v\|_{H_2^\sigma}. \end{aligned}$$

The corollary is proved. \square

The constant $\sigma + 2$ gives the regularity of solution u . To see how smooth the above solution is, we recall

$$\sigma_0 := \gamma_0 + (\varepsilon_0 1_{\gamma=1/2}) = \begin{cases} (2\gamma - 1)/\beta & \text{if } \gamma > 1/2 \\ \varepsilon_0 & \text{if } \gamma = 1/2 \\ 0 & \text{if } \gamma < 1/2. \end{cases}$$

Since $\sigma + 2 = (\sigma + \sigma_0) + (2 - \sigma_0) < -1/2 + (2 - \sigma_0)$, it follows

$$\sigma + 2 < \begin{cases} \frac{3}{2} - \frac{2\gamma-1}{\beta} & \text{if } \gamma > 1/2 \\ \frac{3}{2} & \text{if } \gamma \leq 1/2. \end{cases}$$

Since we are assuming $\sigma + 2 > 0$, we need

$$\gamma < \frac{1}{2} + \frac{3}{4}\beta,$$

which is slightly stronger than (1.9).

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